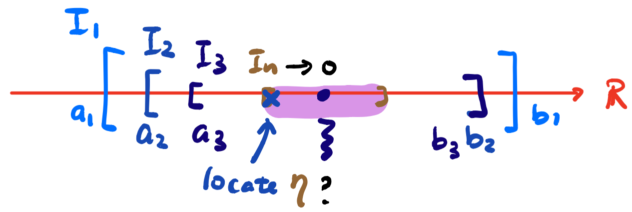
MATH 2050C Lecture on 2/21/2020

Announcement: PS3 due tonight, PS4 posted and due next Fri (Feb 28).

Last lecture: intervals, a characterization of interval and

Thm: (Nested Interval Property)



Let $I_n \subseteq \mathbb{R}$, $n \in \mathbb{N}$ s.t.

(i) Each I_n is a closed and bdd interval, for $n \in \mathbb{N}$.

(ii) Nested: $I_{n+1} \subseteq I_n \quad \forall n \in \mathbb{N}$.

Then, (1) $\displaystyle\bigcap_{n=1}^{\infty} I_n := \{ x \in \mathbb{R} \mid x \in I_n \ \forall n \in \mathbb{N} \} \neq \phi$

(2) If $\inf_n \text{Length}(I_n) = 0$, then $\displaystyle\bigcap_{n=1}^{\infty} I_n = \{\xi\}$ for some $\xi \in \mathbb{R}$.

Proof: Let $I_n = [a_n, b_n]$ for some $a_n < b_n$. $\forall n \in \mathbb{N}$. (by (i))

(ii) $\Rightarrow$ $a_1 \le a_2 \le a_3 \le \cdots \le a_n < b_n \le b_{n-1} \le \cdots \le b_2 \le b_1 \quad \forall n \in \mathbb{N}$

Consider $\mathcal{A} := \{ a_n : n \in \mathbb{N} \} \subseteq \mathbb{R}$,

- $\mathcal{A} \neq \phi$ and $\mathcal{A}$ is bdd above by b_1
- Completeness of $\mathbb{R}$ $\Rightarrow$ $\eta := \sup \mathcal{A} \in \mathbb{R}$ is well-defined.

Claim: $\eta \in \displaystyle\bigcap_{n=1}^{\infty} I_n$ ✓, i.e. $\eta \in I_n = [a_n, b_n]$ for each $n \in \mathbb{N}$.

Pf of Claim: • η is an upper bd of $\mathcal{A}$ $\Rightarrow$ $a_n \le \eta \quad \forall n \in \mathbb{N}$

• Want: $\eta \le b_n \quad \forall n \in \mathbb{N}$ ✓

By Contradiction! Suppose NOT, $\exists\, m \in \mathbb{N}$ s.t. $b_m < \eta$.

Since η is the l.u.b. of $\mathcal{A}$, so b_m cannot be an upper bd.

$\Rightarrow \exists\, k \in \mathbb{N}$ s.t. $b_m < a_k$

- If $m < k$, then $b_k \le b_m < a_k$ Contradiction!
- If $m \ge k$, then $b_m < a_k \le a_m$ Contradiction!

This proves (1). Exercise: Proof of (2). contradiction argument. $\square$

Recall: $\mathbb{N}, \mathbb{Q}$ countable

Corollary: $\mathbb{R}$ is uncountable.

Proof: • Suffices to prove $[0,1]$ is uncountable.

• Suppose NOT, i.e. $[0,1] = \{x_1, x_2, x_3, x_4, \ldots\}$

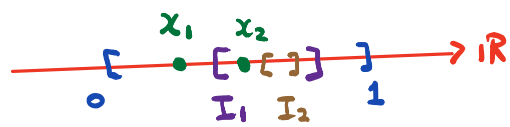
• We will define inductively a seq. of nested, closed, bdd intervals I_n, $n \in \mathbb{N}$.

- Choose $I_1 \subseteq [0,1]$ closed and bdd

  s.t. $x_1 \notin I_1$

- Choose $I_2 \subseteq [0,1]$ closed and bdd

  s.t. $x_2 \notin I_2 \subseteq I_1$



- Choose $I_n \subseteq [0,1]$ closed and bdd

  s.t. $x_n \notin I_n \subseteq I_{n-1}$

• So, I_n is closed, bdd and nest,

N.I.P. $\Rightarrow \exists\, \eta \in \bigcap_{n=1}^{\infty} I_n \neq \emptyset \quad \Rightarrow \quad \eta \in I_n \quad \forall n \in \mathbb{N}$

Thus, by def^n of I_n, $\eta \neq x_n$ for all $n \in \mathbb{N}$. Contradiction: $\eta \in [0,1]$ $\square$

Sequences and their limits (§ 3.1)

GOAL: Define "$\lim_{n\to\infty} x_n = x$", study its properties,

and determine "convergence / divergence" of sequences.

Q: What is a seq. of real numbers?

Def^n: A sequence of real numbers is a function $X: \mathbb{N} \to \mathbb{R}$

denote: $X(1) =: x_1$, $X(2) =: x_2$, …, $X(n) =: x_n$

defining

write: $X = (x_n) = (x_1, x_2, x_3, \ldots, x_n, \ldots\ldots)$

Seq. vs Sets: In general, $(x_n : n \in \mathbb{N}) \neq \{x_n : n \in \mathbb{N}\}$

E.g.) $\left((-1)^n\right) = (-1, 1, -1, 1, \ldots\ldots)$ ordered & infinite

$\{(-1)^n : n \in \mathbb{N}\} = \{-1, 1\}$ unordered & could be finite

Examples: (1) constant seq. $(1, 1, 1, 1, 1, \ldots\ldots)$

(2) geometric seq. $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots\ldots\right) = \left(\frac{1}{2^n}\right)$

(3) arithmetic seq. $(1, 4, 7, 10, 13, \ldots) = (1 + 3(n-1))$

- even no. $(0, 2, 4, 6, 8, \ldots)$
- odd no. $(1, 3, 5, 7, 9, \ldots)$

(4) Fibonacci seq. (inductively / recursively defined)

$x_1 := 1$; $x_2 := 1$

for $n \geq 3$, $x_n := x_{n-1} + x_{n-2}$

$(x_n) = (1, 1, 2, 3, 5, 8, 13, \ldots\ldots)$

Consider $\left(\frac{1}{n}\right) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \Big|_{K} \frac{1}{1000}, \ldots\right)$

Q: How to describe this? Feeling: $\longrightarrow 0$ towards the "end" of the seq.

Def$^{\underline{n}}$: (ε-K definition of limit)

We say (x_n) converges to $x \in \mathbb{R}$, writen:

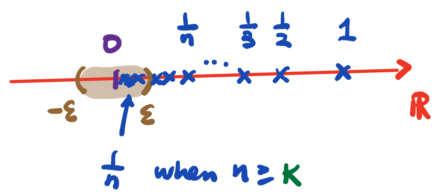
$\lim (x_n) = x$ or $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$

iff $\forall \varepsilon > 0$, $\exists K = K(\varepsilon) \in \mathbb{N}$ s.t.

"small" (ε) "large" (K)

$$|x_n - x| < \varepsilon \quad \text{for any } n \geq K.$$

(i.e. $x - \varepsilon < x_n < x + \varepsilon$ for any $n \geq K$)



Idea: x_n "eventually" gets very "close" to x

$\quad$ "eventually" ~ K, "close" ~ ε

Example: $\boxed{\lim\left(\frac{1}{n}\right) = 0}$

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $K(\varepsilon) \in \mathbb{N}$ s.t. $\frac{1}{K} < \varepsilon$ (i.e. $K > \frac{1}{\varepsilon}$) by Archemedian Property

Want: $\forall\, n \geq K$, we have

$$\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{K} < \varepsilon$$

Terminology: Given a seq. (x_n), we say

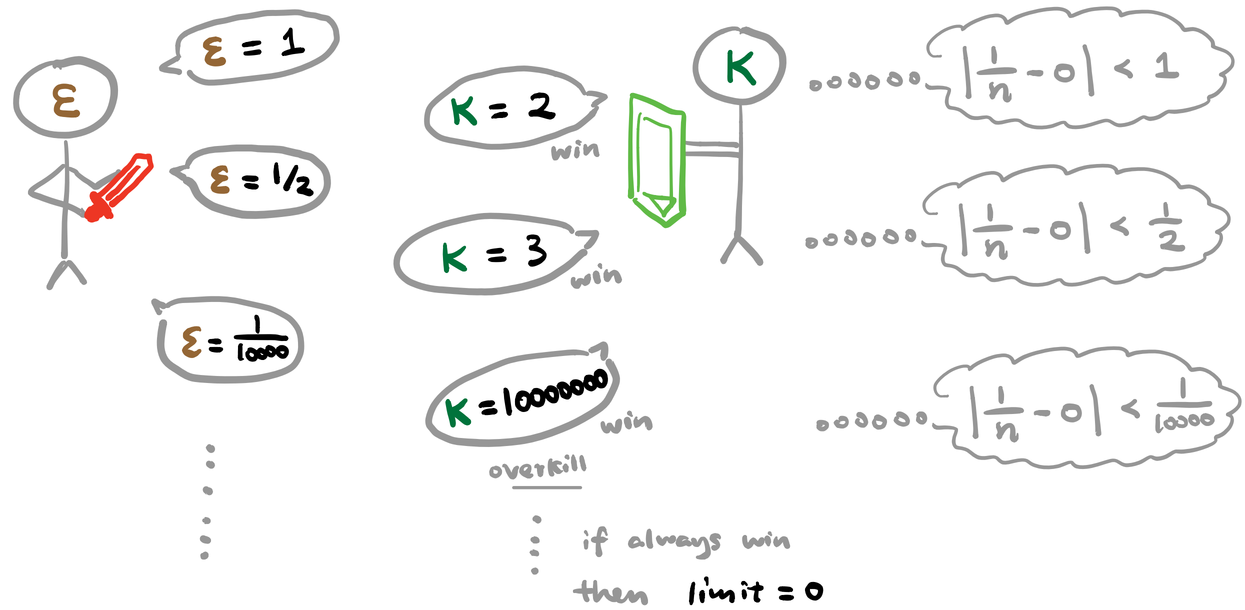
(1) (x_n) is convergent if $\exists\, x \in \mathbb{R}$ s.t. $\lim(x_n) = x$

(2) (x_n) is divergent if it is NOT convergent

i.e. $\cancel{\exists}\, x \in \mathbb{R}$ s.t. $\lim(x_n) = x$

"Dynamical view of limit"

- Recall: $\lim\left(\frac{1}{n}\right) = 0$

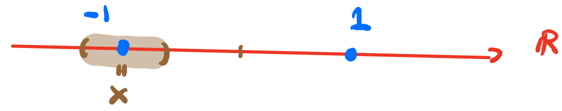


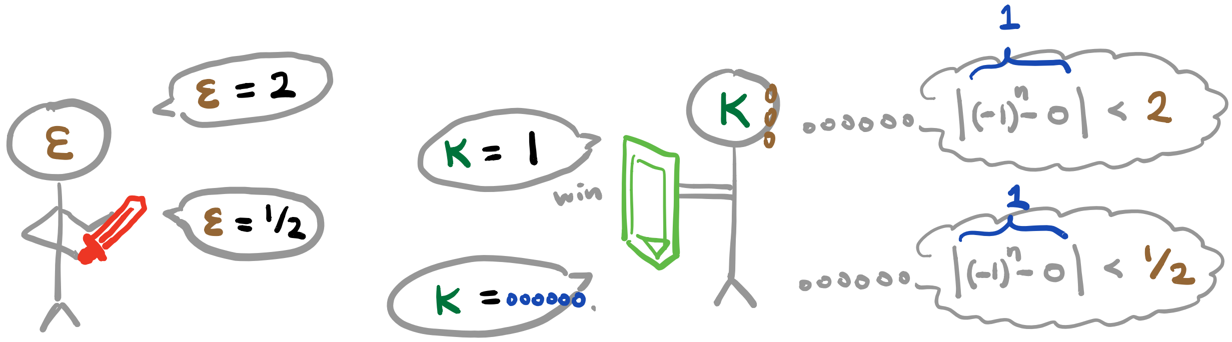
Q: What about the case when limit does exist ?

A divergent example: $(x_n) = ((-1)^n) = (-1, 1, -1, 1, -1, 1, \dots)$

Claim: This seq. is divergent, i.e. $x_n \not\to x$ for any $x \in \mathbb{R}$.

Q: Is $\lim(x_n) = 0$? NO.





LOSE! $\Rightarrow \lim(x_n) \neq 0$.

Q: Is $\lim(x_n) = 1$? NO.